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# An Optimal Transport Approach for Selecting a Representative Subsample with Application in Efficient Kernel Density Estimation

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## ABSTRACT

Subsampling methods aim to select a subsample as a surrogate for the observed sample. Such methods have been used pervasively in large-scale data analytics, active learning, and privacy-preserving analysis in recent decades. Instead of model-based methods, in this article, we study model-free subsampling methods, which aim to identify a subsample, that is, not confined by model assumptions. Existing model-free subsampling methods are usually built upon clustering techniques or kernel tricks. Most of these methods suffer from either a large computational burden or a theoretical weakness. In particular, the theoretical weakness is that the empirical distribution of the selected subsample may not necessarily converge to the population distribution. Such computational and theoretical limitations hinder the broad applicability of model-free subsampling methods in practice. We propose a novel model-free subsampling method by using optimal transport techniques. Moreover, we develop an efficient subsampling algorithm, that is, adaptive to the unknown probability density function. Theoretically, we show the selected subsample can be used for efficient density estimation by deriving the convergence rate for the proposed subsample kernel density estimator. We also provide the optimal bandwidth for the proposed estimator. Numerical studies on synthetic and real-world datasets demonstrate the performance of the proposed method is superior.

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## KEYWORDS

Density estimation; Inverse transform sampling; Optimal transport; Star discrepancy; Subsampling

## 1. Introduction

A subsampling problem can be described as follows: given a  $d$ -dimensional sample  $\{\mathbf{x}_i\}_{i=1}^n$  generated from an unknown probability distribution, the goal is to take a subsample  $\{\mathbf{x}_i^*\}_{i=1}^r$ ,  $r \ll n$ , as a surrogate for the original sample. In recent decades, the subsampling problem has drawn great attention in machine learning, statistics, and computer science. For example, subsampling methods are used pervasively in optimal design/active learning problems, where in a large sample of unlabeled data, the goal is to select an informative subsample to label (Settles 2012). Consider privacy-preserving analysis as another example. In some applications, subsampling methods have the potential to enhance data security (Nissim, Raskhodnikova, and Smith 2007; Li, Qardaji, and Su 2012). Specifically, a carefully selected subset of data can reveal little confidential information (Shu, Yao, and Bertino 2015). Last but not least, subsampling methods are also widely applied in algorithm design to alleviate the computational burden in large-scale data analysis (Tsai et al. 2015; Zhou et al. 2017).

Many existing subsampling methods are model-based methods, which assume predictors and responses, if any, follow a postulated model. These methods aim to select an informative subsample that benefits model-fitting and prediction. Various models have been considered in subsampling problems, including

linear regression (Drineas, Kannan, and Mahoney 2006; Drineas et al. 2011; Ma, Mahoney, and Yu 2014, 2015; Ma and Sun 2015; Wang, Yu, and Singh 2017; Meng et al. 2017; Zhang, Xie, and Ma 2018; Ma et al. 2020; Li and Meng 2020), generalized linear regression (Wang, Zhu, and Ma 2018; Ai et al. 2021a; Yu et al. 2022),  $l_p$  regression (Dasgupta et al. 2009), quantile regression (Ai et al. 2021b), streaming time series model (Xie et al. 2019), Gaussian mixture model (Feldman, Faulkner, and Krause 2011), nonparametric regression (Meng et al. 2020b, 2022), among others (Bardenet, Doucet, and Holmes 2017; Quiroz et al. 2019; Yu and Wang 2022). While model-based subsampling methods have already yielded impressive achievements, the key to the success of these methods highly depends on the correct model specification. Nevertheless, in practice, model specification is a trial and error process, and a postulated model for the data could be misspecified. For example, in supervised learning, we start with a high dimensional model with numerous features; and by using model selection, we may end up with a low dimensional model with parsimonious features. In another instance, we may start with a linear regression model for a continuous response; and by discretizing the response, we may end up with a classification model. Model-based subsampling methods, however, may result in subsamples hampering such dynamic processes of model specification (Tsao and Ling 2012). Consequently, in scenarios when the model may be misspecified or in the stage

of exploratory analysis, more preferred methods are model-free subsampling methods, which can identify a subsample, that is, not confined by model assumptions.

Recently, there have been emerging model-free subsampling methods, which aim to select a representative subsample that can capture the overall patterns of the observed sample. These methods can be divided into two classes: clustering-based approaches and kernel-based approaches. Clustering-based approaches, which are usually used in unsupervised learning methods, include  $k$ -medoids method (Kaufman and Rousseeuw 1987; Park and Jun 2009),  $k$ -center method (Feder and Greene 1988), and Wasserstein barycenter method (Agueh and Carlier 2011; Cuturi and Doucet 2014). The  $k$ -medoids method is closely related to the  $k$ -means algorithm, and the  $k$ -center method is used extensively in fast multipole methods (Greengard and Strain 1991; White et al. 1994; Yang et al. 2003; Lee and Gray 2009). The Wasserstein barycenter method aims to find the barycenter of a set of empirical probability measures under the optimal transport metric, and such a barycenter itself can be regarded as a representative subsample. Despite wide applications of these subsampling methods, the empirical distributions of the selected subsamples, yielded by these clustering-based approaches, may not resemble the probability distribution of the original sample. That is, as the subsample size increases, the probability distributions of the subsample identified by these methods may not necessarily converge to the true probability distribution. To address such a limitation, researchers developed kernel-based approaches, which aim to select a subsample that can effectively approximate the population distribution. These approaches include the kernel herding method (Chen and Zhang 2014), the coresets for kernel density estimation (Phillips 2013; Zheng et al. 2013, 2017), and the support point method (Mak and Joseph 2018). Despite the theoretical benefits, one limitation of these kernel-based approaches is that they may result in a large computational burden in large-scale data analysis.

To overcome the computational and theoretical limitations of the aforementioned methods, we propose a novel model-free subsampling method, that is, computationally efficient and enjoys nice theoretical properties. The proposed method combines the techniques of optimal transport and space-filling designs. In particular, we first transform the observed sample to be uniformly distributed on a hypercube using optimal transport techniques (Villani 2008; Peyré et al. 2019), then select a set of data points that can effectively represent the uniform distribution using space-filling designs (Owen 2003; Fang, Li, and Sudjianto 2005). The desired subsample is the one corresponding to the selected data points. The idea is analogous to an inverse procedure of the inverse transform sampling technique, which transforms a uniformly distributed sample to a sample that follows an arbitrary probability density function. Theoretically, we show the proposed subsample kernel density estimator converges to the true probability density function under mild conditions. Moreover, we show the proposed estimator converges faster than the estimator based on a randomly selected subsample, suggesting the proposed method can be used for efficient density estimation. We also provide the optimal bandwidth for the proposed estimator. Numerically, using projection-based optimal transport methods (Pitje, Kokaram, and Dahyot 2005; Rabin et al. 2011), the

computational cost for the proposed method is at the order of  $O(n \log(n) d^2)$  for a  $d$ -dimensional sample of size  $n$ . The proposed method thus is scalable to datasets with large  $n$  and moderate  $d$ . Numerical studies on synthetic and real-world datasets demonstrate the superior performance of the proposed method in comparison with mainstream competitors. The proposed method is implemented in an R package, named SPARTAN.

## 2. Preliminaries

### 2.1. Star Discrepancy and Space-Filling Designs

The proposed method is developed upon the notion of star discrepancy, which is a classical metric that measures the discrepancy between a set of discrete data points and the uniform distribution on the unit hypercube  $[0, 1]^d$ , denoted by  $U[0, 1]^d$  (Niederreiter 1992; Fang and Wang 1993; Fang, Li, and Sudjianto 2005). Let  $1\{\cdot\}$  be the indicator function and  $\mathbf{a} = (a_1, \dots, a_d) \in [0, 1]^d$  be a vector. Let  $[\mathbf{0}, \mathbf{a}] = \prod_{j=1}^d [0, a_j]$  be a hyper-rectangle and  $\mathcal{U}_r = \{\mathbf{u}_i\}_{i=1}^r$  be a set of  $r$  data points in  $[0, 1]^d$ . We introduce the definition of the star discrepancy in the following.

**Definition 1.** Given  $\mathcal{U}_r$  and a hyper-rectangle  $[\mathbf{0}, \mathbf{a}]$ ,  $\mathbf{a} \in [0, 1]^d$ , the corresponding local discrepancy is defined as,  $D(\mathcal{U}_r, \mathbf{a}) = |\frac{1}{r} \sum_{i=1}^r 1\{\mathbf{u}_i \in [\mathbf{0}, \mathbf{a}]\} - \prod_{j=1}^d a_j|$ . The star discrepancy is defined as

$$D^*(\mathcal{U}_r) = \sup_{\mathbf{a} \in [0, 1]^d} D(\mathcal{U}_r, \mathbf{a}).$$

Definition 1 suggests a set of data points  $\mathcal{U}_r$ , which can effectively represent  $U[0, 1]^d$ , has a small value of  $D^*(\mathcal{U}_r)$ , and vice versa. There exist methods that generate design points via directly minimizing the star discrepancy, and these methods are called uniform design methods (Fang, Li, and Sudjianto 2005). Despite wide applications, most of these methods are computationally expensive and are not scalable to a design with a large number of points. To alleviate such a computational burden, methods yielding a set of design points with a relatively small star discrepancy could be used as alternatives for uniform design methods. These alternatives include space-filling design methods (Wu and Hamada 2011; Fang, Li, and Sudjianto 2005) and low-discrepancy sequences (Owen 2003; Lemieux 2009; Dick, Kuo, and Sloan 2013; Leobacher and Pillichshammer 2014). The former aims to generate a set of design points that spread out over the domain as uniformly as possible. The latter sequentially generates the design points, which achieve an asymptotically fast decay rate respecting the star discrepancy. Consequently, these methods provide powerful tools to generate a set of representative design points in terms of  $U[0, 1]^d$ .

We now discuss the theoretical property of space-filling designs and low-discrepancy sequences in terms of the star discrepancy (Owen 2003). For a Sobol sequence  $\mathcal{S}_r = \{\mathbf{s}_i\}_{i=1}^r$ , a representative of low-discrepancy sequences,  $D^*(\mathcal{S}_r)$  converges to zero at the rate of  $O(\log(r)^d/r)$ . In other words, the convergence rate of  $D^*(\mathcal{S}_r)$  is of the order  $O(r^{-(1-\delta)})$  for an arbitrary small  $\delta > 0$  and fixed  $d$ , as  $r$  goes to infinity. For comparison, when a set of data points  $\mathcal{X}_r = \{\mathbf{x}_i\}_{i=1}^r$  is randomly

generated from  $U[0, 1]^d$ , the convergence rate of  $D^*(\mathcal{X}_r)$  is of the order  $O((\log \log(r)/r)^{1/2})$ , which is much slower than  $O(r^{-(1-\delta)})$  (Chung 1949). By adopting a method which is no worse than the Sobol sequence, in this article, we always assume the star discrepancy  $D^*(\mathcal{S}_r)$  converges to zero with the rate  $O(r^{-(1-\delta)})$ . There also exist some space-filling designs that can achieve a potentially faster convergence rate in terms of star discrepancy (Fang, Li, and Sudjianto 2005).

Using space-filling design techniques, we propose a simple algorithm to select a representative subsample from a sample, that is, generated from  $U[0, 1]^d$ . Let  $\{\mathbf{u}_i\}_{i=1}^n$  be such a sample. The proposed algorithm, summarized in Algorithm 1, combines space-filling design techniques and the one-nearest-neighbor approximation.

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**Algorithm 1** Select a representative subsample from a sample generated from  $U[0, 1]^d$ .

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- Step 1. Generate a set of space-filling design points  $\{\mathbf{s}_i\}_{i=1}^r \in [0, 1]^d$
  - Step 2. For  $i = 1$  to  $r$ 
    - Select the nearest neighbor for  $s_i$  from  $\{\mathbf{u}_i\}_{i=1}^n$  using the Euclidean distance
    - Let  $\mathbf{u}_i^*$  be the selected data point
  - Step 3. The final subsample is given by  $\mathcal{U}_r^* = \{\mathbf{u}_i^*\}_{i=1}^r$
- 

**Lemma 1** below, which is first stated in Meng et al. (2020b), characterizes the approximation error of the subsample selected by Algorithm 1. This lemma suggests the selected subsample can effectively approximate the design points in the sense that their corresponding star discrepancies are almost at the same order under certain conditions.

**Lemma 1.** Let  $\mathcal{S}_r = \{\mathbf{s}_i\}_{i=1}^r \in [0, 1]^d$  be a set of design points which satisfy  $D^*(\mathcal{S}_r) = O(r^{-(1-\delta)})$  for any arbitrary small  $\delta > 0$ , as  $r \rightarrow \infty$ . Suppose  $d$  is fixed, when  $r = O(n^{1/d})$ , as  $n \rightarrow \infty$ , we have  $D^*(\mathcal{U}_r^*) = O_p(r^{-(1-\delta)})$ .

Algorithm 1 can be extended to the case that the cumulative distribution function  $F$  of the samples is nonuniform when  $d = 1$ . The idea is analogous to the classical inverse transform sampling method (Devroye 1986; Mosegaard and Tarantola 1995). Let  $\{x_i\}_{i=1}^n \in \mathbb{R}$  be the observed sample, we first calculate  $\{F(x_i)\}_{i=1}^n$ , from which, we then select a subsample  $\{F(x_i^*)\}_{i=1}^r$  using Algorithm 1. Notice that the transformed sample is uniformly distributed on  $[0, 1]$ ; thus, the selected subsample is relatively representative of  $U[0, 1]$ . Finally, the desired subsample is given by  $\{x_i^*\}_{i=1}^r$ . Although this simple strategy works well in practice, a limitation of such a strategy is that it is inapplicable when  $d \geq 2$ <sup>1</sup>. To overcome the limitation, we introduce the optimal transport map, which serves as a surrogate for  $F$  in multivariate cases. This idea is similar to the one in Chernozhukov et al. (2017), where the authors used the optimal transport map to extend the concepts of quantiles and

ranks from one-dimensional samples to multivariate samples. Analogously, in this article, we use the optimal transport map to extend the technique of inverse transform sampling from one-dimensional cases to high-dimensional cases.

## 2.2. Optimal Transport Maps

Optimal transport maps have been extensively used as a standard technique to transform one probability distribution to another. Recently, such maps have received a significant attention in machine learning and computer science (Ferradans et al. 2014; Rabin, Ferradans, and Papadakis 2014; Su et al. 2015; Courty et al. 2017; Meng et al. 2020a; Peyré et al. 2019), due to its close relationship with generative models, including generative adversarial nets (Goodfellow et al. 2014), the “decoder” network in variational autoencoders (Kingma and Welling 2013), among others.

Instead of introducing the general definition of the optimal transport map, we now present a specific map of our interest, and we refer to Villani (2008); Peyré et al. (2019); Zhang, Zhong, and Ma (2021) for more details. Let  $u$  be the uniform probability distribution on  $[0, 1]^d$ . Let  $p_X$  and  $\Omega \subseteq \mathbb{R}^d$  be the probability distribution and the domain of the random variable  $X$ , respectively. Let  $\#$  be the push-forward operator, such that for all measurable  $B \subset \Omega$ , we have  $\phi_{\#}(p_X)(B) = p_X(\phi^{-1}(B))$ . Among all the maps  $\phi : \Omega \rightarrow [0, 1]^d$  such that  $\phi_{\#}(p_X) = u$  and  $\phi_{\#}^{-1}(u) = p_X$ , the optimal transport map  $\phi^*$  of our interest is the one that minimizes the  $L_2$  cost,  $\int_{\Omega} \|X - \phi(X)\|^2 dp_X$ , where  $\|\cdot\|$  denotes the Euclidean norm. We focus on  $L_2$  cost in this article for simplicity and it is possible to consider other costs as long as the optimal transport map exists. For the  $L_2$  cost, as a special case, when  $\Omega = \mathbb{R}$  and  $d = 1$ , it is known that  $\phi^*$  is equivalent to the cumulative distribution function  $F$  (Villani 2008). This fact motivates us to use the  $\phi^*$  as a surrogate for  $F$  in high-dimensional cases.

To obtain the desired optimal transport map that maps the observed sample to be uniformly distributed on  $[0, 1]^d$ , we propose to first generate a synthetic sample from  $U[0, 1]^d$ , then calculate the optimal transport map from the observed sample to the synthetic sample. One can use the auction algorithm or the refined auction algorithm to calculate such a map (Bertsekas 1992; Schuhmacher et al. 2020). Despite the effectiveness, the auction algorithm has an average computational cost of the order  $O(n^2)$ , and thus it may incur an enormous computational cost when  $n$  is large. To alleviate the computational burden, in practise, we propose to approximate the optimal transport map  $\phi^*$  using projection-based methods (Pitié, Kokaram, and Dahyot 2007; Bonneel et al. 2015; Rabin et al. 2011; Meng et al. 2019; Zhang et al. 2022). These methods tackle the problem of estimating a  $d$ -dimensional optimal transport map iteratively by breaking down the problem into a series of subproblems. Each of the subproblems involves finding a one-dimensional optimal transport map between the projected samples, and such a subproblem can be easily solved through sorting algorithms.

## 3. Main Algorithm

We develop a novel subsampling method named SPARTAN, which integrates space-filling design techniques and optimal

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<sup>1</sup>One exception is that when all the covariates of the sample are independent with each other, in which case one can directly calculate the multivariate cumulative distribution function as the product of all the one-dimensional marginal cumulative distribution function. Nevertheless, independent covariates are rarely the case in practice.

transport methods. The proposed method works as follows. First, we transform the observed sample, denoted by  $\{\mathbf{x}_i\}_{i=1}^n$ , to be uniformly distributed on  $[0, 1]^d$ . We achieve this goal by using the empirical optimal transport map. Here, the empirical optimal transport map is also called the optimal matching between two discrete distributions, such that each of them have  $n$  atoms and each atom has weight  $1/n$ . We use such an empirical optimal transport map as a surrogate of the optimal transport map between the underlying population density function of the observed sample and the uniform distribution. We then select a set of data points of size  $r$  from the transformed sample using [Algorithm 1](#). The subsample corresponding to the selected data points is the final output. We summarize the algorithm below.

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**Algorithm 2** Space-filling after optimal transport (SPARTAN)

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- Step 1.* Generate a synthetic random sample  $\{\mathbf{u}_i\}_{i=1}^n$  from  $U[0, 1]^d$
- Step 2.* Calculate the empirical optimal transport map, denoted by  $\hat{\phi}$ , that maps the observed sample  $\{\mathbf{x}_i\}_{i=1}^n$  to the synthetic sample  $\{\mathbf{u}_i\}_{i=1}^n$
- Step 3.* Calculate the transformed sample  $\{\hat{\phi}(\mathbf{x}_i)\}_{i=1}^n$
- Step 4.* Select a set of data points  $\{\hat{\phi}(\mathbf{x}_i^*)\}_{i=1}^r$  from  $\{\hat{\phi}(\mathbf{x}_i)\}_{i=1}^n$  using [Algorithm 1](#)
- Step 5.* The final subsample is given by  $\{\mathbf{x}_i^*\}_{i=1}^r$ .
- 

[Figure 1](#) illustrates [Algorithm 2](#) using a toy example. A two-dimensional synthetic sample of size 1000, marked as grey dots, is shown in [Figure 1\(a\)](#). We first transform the sample to be uniformly distributed on  $[0, 1]^2$  using the projection pursuit Monge map method (Meng et al. 2019), shown in [Figure 1\(b\)](#). We then generate 32 design points using a space-filling design method (Owen 2003; Fang, Li, and Sudjianto 2005). The design points are marked as triangles in [Figure 1\(c\)](#). Next, for each design point, we search for its nearest neighbor, labeled as black dots in [Figure 1\(c\)](#). Finally, the subsample corresponding to the selected data points, marked as black dots in [Figure 1\(d\)](#), gives the desired subsample.

The computational cost for [Algorithm 2](#) mainly incurs in Step 2 and Step 4. In particular, we use a projection-based method to approximate the desired optimal transport map in Step 2, requiring a computational cost of the order  $O(n \log(n) d^2)$  (Pitié, Kokaram, and Dahyot 2007; Bonneel et al. 2015; Meng et al. 2019). Step 4 includes two sub-steps: generating the design points and searching the corresponding nearest neighbors. The design points can be generated beforehand; thus, the computation time for generating these points is not considered here. For searching the nearest neighbors, we opt to use the  $k$ -d tree method, whose computation cost is at the order of  $O(n \log(n))$  (Bentley 1975; Wald and Havran 2006). In sum, the overall computational complexity for [Algorithm 1](#) is at the order of  $O(n \log(n) d^2)$ .

[Figure 2](#) visualizes the subsamples (black dot) selected by the proposed method (lower row) compared with the subsamples selected by the random subsampling method (upper row). The two-dimensional samples (grey dots) are generated from three different distributions: the standard Gaussian distribution (left column), a mixture Gaussian distribution (middle column), and

a mixture beta distribution (right column). From plots in the left column, one can observe that the randomly selected subsample is far from symmetric. From plots in the middle and the right columns, one can see that some peaks in the probability distribution are largely overlooked by the random subsampling method. We observe that the subsamples identified by the proposed method have a more robust and appealing visual representation of the corresponding probability distribution in all the cases.

## 4. Theoretical Results

In this section, we study the theoretical properties of the subsamples obtained in [Algorithm 2](#). In particular, we develop an asymptotic theory concerning the rates of convergence of the estimated density to the true density as the sample size goes to infinity. The rates are calculated in terms of the pointwise mean squared error (MSE) that defined as  $\text{MSE}(\hat{p}(\mathbf{z})) = E\{\hat{p}(\mathbf{z}) - p(\mathbf{z})\}^2$ , where  $\mathbf{z} \in \mathbb{R}^d$ ,  $\hat{p}$  is the density estimator and  $p$  is the true density. The density is estimated using the widely used kernel density estimation method. Throughout this article, we consider the Gaussian kernel. The extension of the main theorem to other kernel functions is straightforward, as long as such a kernel function satisfies some regularity conditions, which are relegated to the Supplementary Material. A more in-depth discussion on different choices of kernel functions can be found in Scott (2015). To avoid trivial cases, we consider the case that  $d \geq 2$  in this section. Without loss of generality, we assume the points  $\{\mathbf{x}_i\}$  are distinct, and the points  $\{\mathbf{u}_i\}$  are distinct. In such cases, the optimal transport map in Step 2 of [Algorithm 2](#) is a one-to-one map from  $\{\mathbf{x}_i\}_{i=1}^n$  to  $\{\mathbf{u}_i\}_{i=1}^n$ . Let  $p$  be the probability density function to be estimated. Two widely used regularity conditions for  $p$  are required in kernel density estimation,

- Condition (a).  $\partial^2 p(\mathbf{z}) / \partial z_j^2$  is absolutely continuous, for  $j = 1, \dots, d$ ;
- Condition (b).  $\partial^3 p(\mathbf{z}) / \partial z_j^3$  is square-integrable, for  $j = 1, \dots, d$ .

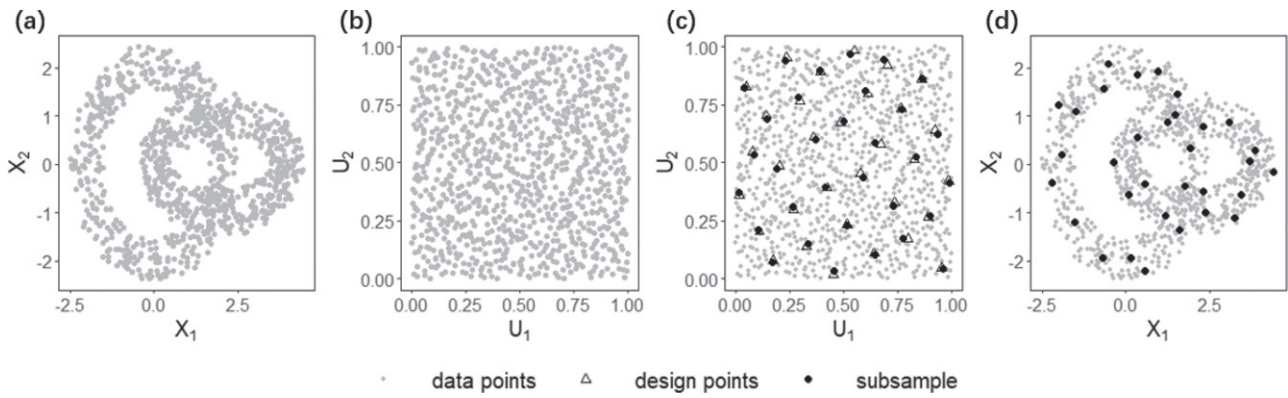
Let  $\mathbf{X} \in \mathbb{R}^{n \times d}$  be the sample matrix, where the  $(i, j)$ th element is  $x_{ij}$ , and  $\mathbf{X}^* \in \mathbb{R}^{r \times d}$  be the subsample matrix, where the  $(i, j)$ th element is  $x_{ij}^*$ . Let  $h > 0$  be the bandwidth and  $\mathcal{K} : \mathbb{R} \rightarrow \mathbb{R}$  be a kernel function. For any  $\mathbf{z} \in \mathbb{R}^d$ , the full-sample product kernel density estimator can be written as

$$\hat{p}(\mathbf{z}) = \sum_{i=1}^n \left[ \prod_{j=1}^d \mathcal{K}\{(z_j - x_{ij})/h\} / h \right] / n. \quad (1)$$

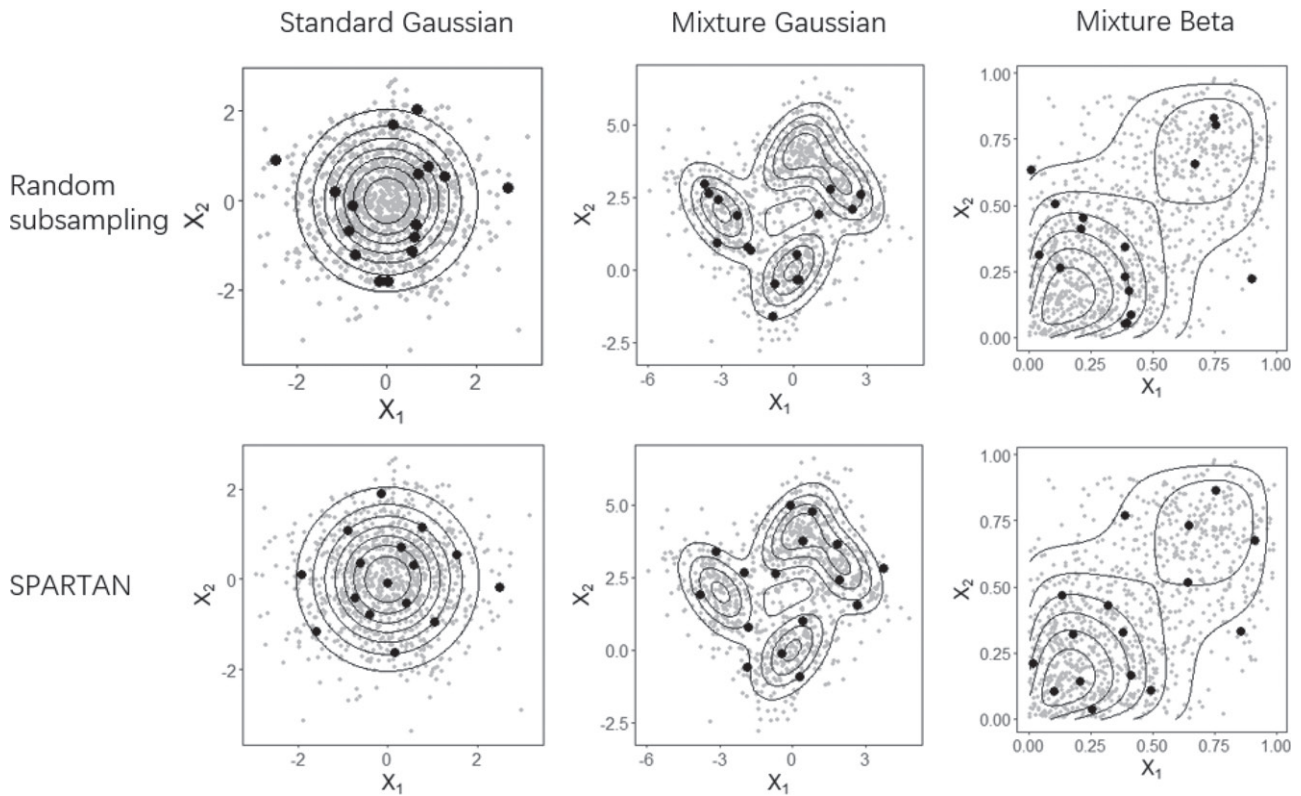
Equation (1) can be generalized to a more general multivariate kernel density estimator. In particular, for a  $d \times d$  nonsingular bandwidth matrix  $\mathbf{H}$  and a multivariate kernel function  $\mathcal{K} : \mathbb{R}^d \rightarrow \mathbb{R}$ , a general multivariate kernel estimator can be written as

$$\hat{p}_{\text{general}}(\mathbf{z}) = \frac{1}{n|\mathbf{H}|} \sum_{i=1}^n [\mathcal{K}\{\mathbf{H}^{-1}(\mathbf{z} - \mathbf{x}_i)\}]. \quad (2)$$

It is apparent that Equation (2) is equivalent to Equation (1) when  $\mathbf{H} = h \cdot \mathbf{I}_d$ , where  $\mathbf{I}_d$  is the identity matrix. Let  $\mathcal{K}$  be the Gaussian kernel in Equation (2), it is equivalent to choose



**Figure 1.** Illustration for Algorithm 2. The two-dimensional sample, marked as gray dots in panel (a), is first transformed to be uniformly distributed on  $[0, 1]^2$ , shown in panel (b). We then generate a set of space-filling design points, marked as triangles, and search for the nearest neighbor for each of them, marked by black dots in panel (c). Panel (d) shows the subsample corresponding to the selected data points.



**Figure 2.** Subsamples (black dots) selected by the proposed method (lower) versus randomly selected subsamples (upper). Contours (black) are superimposed. One can observe the proposed method selects subsamples that have more appealing visual representation of the corresponding population.

$\mathcal{K} = \mathcal{N}(\mathbf{0}, \Sigma)$  with  $\mathbf{H} = \mathbf{I}_d$ , or to choose  $\mathcal{K} = \mathcal{N}(\mathbf{0}, \mathbf{I}_d)$  with  $\mathbf{H} = \Sigma^{1/2}$  in Equation (1). Consequently, with a properly chosen kernel function, one can reformulate a general multivariate kernel estimator to a product kernel density estimator. We thus only focus on the product kernel density estimator in this section without loss of generality.

Analogous to Equation (1), the density estimator  $\hat{p}^*(\mathbf{z})$  that computed from the subsample can be written as

$$\hat{p}^*(\mathbf{z}) = \sum_{i=1}^r \left[ \prod_{j=1}^d K \left\{ \frac{z_j - x_{ij}^*}{h} \right\} / h \right] / r.$$

We derive the convergence rate for the mean squared error for the proposed subsample estimator. The results are summarized in Theorem 1 below, and the proof is relegated to Appendix.

**Theorem 1.** Suppose  $p$  satisfies Conditions (a) and (b). Moreover, suppose  $p$  has a compact convex domain  $\Omega \subset \mathbb{R}^d$ , and there exists a constant  $c \geq 1$  for which  $c^{-1} \leq p(\mathbf{x}) \leq c$  for any  $\mathbf{x} \in \Omega$ . When  $d \geq 2$ ,  $r = O(n^{1/d})$ , as  $n \rightarrow \infty$  and  $h \rightarrow 0$ , for any arbitrary small  $\delta > 0$ , we have

$$\text{MSE}(\hat{p}^*(\mathbf{z})) = O\left(\frac{1}{r^{2(1-\delta)}h^{d+2}}\right) + O(h^4).$$

In particular, if  $h = O(r^{-2(1-\delta)/(d+6)})$ , we have

$$\text{MSE}(\hat{p}^*(\mathbf{z})) = O(r^{-8(1-\delta)/(d+6)}). \tag{3}$$

Theorem 1 shows the proposed subsample estimator converges to the true probability density function. Moreover,

**Theorem 1** indicates the proposed subsampling method can be used for efficient density estimation. Specifically, let  $\mathbf{X}^+ \in \mathbb{R}^{r \times d}$  be a randomly selected subsample matrix, and  $\hat{p}^+(z)$  be the corresponding subsample estimator. According to Theorem 6.4 of Scott (2015), as  $r = o(n)$  and  $n \rightarrow \infty$ , when  $h = O(r^{-1/(4+d)})$ ,  $\text{MSE}(\hat{p}^+(z))$  achieves the optimal convergence rate  $O(r^{-4/(d+4)})$  for any  $z \in \Omega$ . Such a convergence rate is much slower than the convergence rate in Equation (3). Consequently, **Theorem 1** indicates one can approximate the probability density function  $p$  more efficiently using the proposed subsample kernel density estimator, compared with the counterpart based on a randomly selected subsample.

Consider the bandwidth  $h$ , or generally, the bandwidth matrix  $\mathbf{H} \in \mathbb{R}^{d \times d}$ . In practice, one can determine the value of  $\mathbf{H}$  through the plug-in approach or the cross-validation approach (Duong and Hazelton 2003; Chacón and Duong 2010; Scott 2015). One limitation of these approaches, however, is that they may result in a computational burden for the sample with moderate or large  $n$ . To combat the computational burden, we opt to determine the value of  $\mathbf{H}$  using the general Scott's rule (Scott 2015), which suggests to use  $\mathbf{H} = r^{-1/(d+4)} \times \hat{\Sigma}^{1/2}$  for a subsample kernel density estimator that based on a subsample of size  $r$ . Here,  $\hat{\Sigma}$  is the empirical variance-covariance matrix for the observed sample. Analogously, as suggested by **Theorem 1**, we also consider using  $\mathbf{H} = r^{-2/(d+6)} \times \hat{\Sigma}^{1/2}$  for the proposed estimator. Consider the essential condition in **Theorem 1**, which requires the domain of  $p$  to be compact convex. Empirically, we find the proposed estimator still works reasonably well when such a condition does not hold, as shown in the following section.

## 5. Simulation Results

To evaluate the proposed subsampling method, we compare it with three mainstream competitors in terms of the estimation accuracy of the kernel density estimator. The competitors include the uniform subsampling method, also called the random subsampling method, the  $k$ -medoids method, and the support point method (Mak and Joseph 2018). We use the projection-pursuit Monge map method (Meng et al. 2019) for approximating the optimal transport map in Algorithm 2. All the methods are implemented in  $\mathbb{R}$ , and all the parameters are set as default.

For each subsampling method, we first calculate the subsample kernel density estimator  $\hat{p}(x)$ , then evaluate the accuracy of which using the Hellinger distance (Li, Yang, and Wong 2016), defined as  $1 - \sum_{i=1}^n \sqrt{\hat{p}(x_i)/p(x_i)}/n$ , where  $\{x_i\}_{i=1}^n$  is an independent testing dataset generate from the same probability density function as the training sample. Empirically, we find other metrics, like the mean squared error considered in **Theorem 1**, also yield similar performance. For the kernel density estimator, we use the Gaussian kernel and the general Scott's rule (Scott 2015) to determine the bandwidth matrix. In particular, for all the subsample estimator, the bandwidth matrix  $\mathbf{H} = r^{-1/(d+4)} \times \hat{\Sigma}^{1/2}$ , where  $\hat{\Sigma}$  is the empirical variance-covariance matrix. For the proposed method, we also consider the cases that  $\mathbf{H} = r^{-2/(d+6)} \times \hat{\Sigma}^{1/2}$ , according to **Theorem 1**. The standard errors are calculated through a hundred replicates.

In each replicate, we generate a synthetic training sample with  $n = 10^4$  from  $d = \{2, 5, 10, 20\}$  and each of the following three probability density functions,

- D1: A Gaussian distribution  $\mathcal{N}(\mathbf{0}, \Sigma)$ , where  $\Sigma_{ij} = 0.5^{|i-j|}$ ,  $i, j = 1, \dots, d$ ;
- D2: A mixture Gaussian distribution  $\mathcal{N}(\mathbf{1}, \Sigma)/4 + \mathcal{N}(-\mathbf{1}, \Sigma)/4 + \mathcal{N}(\mathbf{0}, \Sigma)/2$ , where  $\Sigma = 0.8^{|i-j|}$ ,  $i, j = 1, \dots, d$ .
- D3: A mixture  $t$ -distribution, whose degree-of-freedom equals 8, 10, and 12,  $t(\mathbf{0}, \Sigma, 8)/3 + t(\mathbf{0}, \Sigma, 10)/3 + t(\mathbf{0}, \Sigma, 12)/3$ , where  $\Sigma = 0.8^{|i-j|}$ ,  $i, j = 1, \dots, d$ .

**Figure 3** shows the Hellinger distance versus different  $r$  under various settings. Each row represents a particular data distribution D1–D3, and each column represents a particular  $d$ . We use crosses to denote the uniform subsampling method (UNIF), hollow circles to denote the K-medoids method (KM), hollow triangles to denote the support point method (SP), solid circles to denote the proposed method (SPARTAN), and solid triangles to denote the proposed method with  $\mathbf{H} = r^{-2/(d+6)} \times \hat{\Sigma}^{1/2}$  (SPARTAN\*).

Three significant observations can be made from **Figure 3**. We first observe that the K-medoids method performs worse than the uniform subsampling method in almost all cases. Moreover, the support point method outperforms the uniform subsampling method in all cases. We also observe the Hellinger distance yielded by these two methods do not converge to zero in some cases. Such an observation can be attributed to the fact that the probability distribution of the subsample identified by these two methods may not necessarily converge to the true probability distribution.

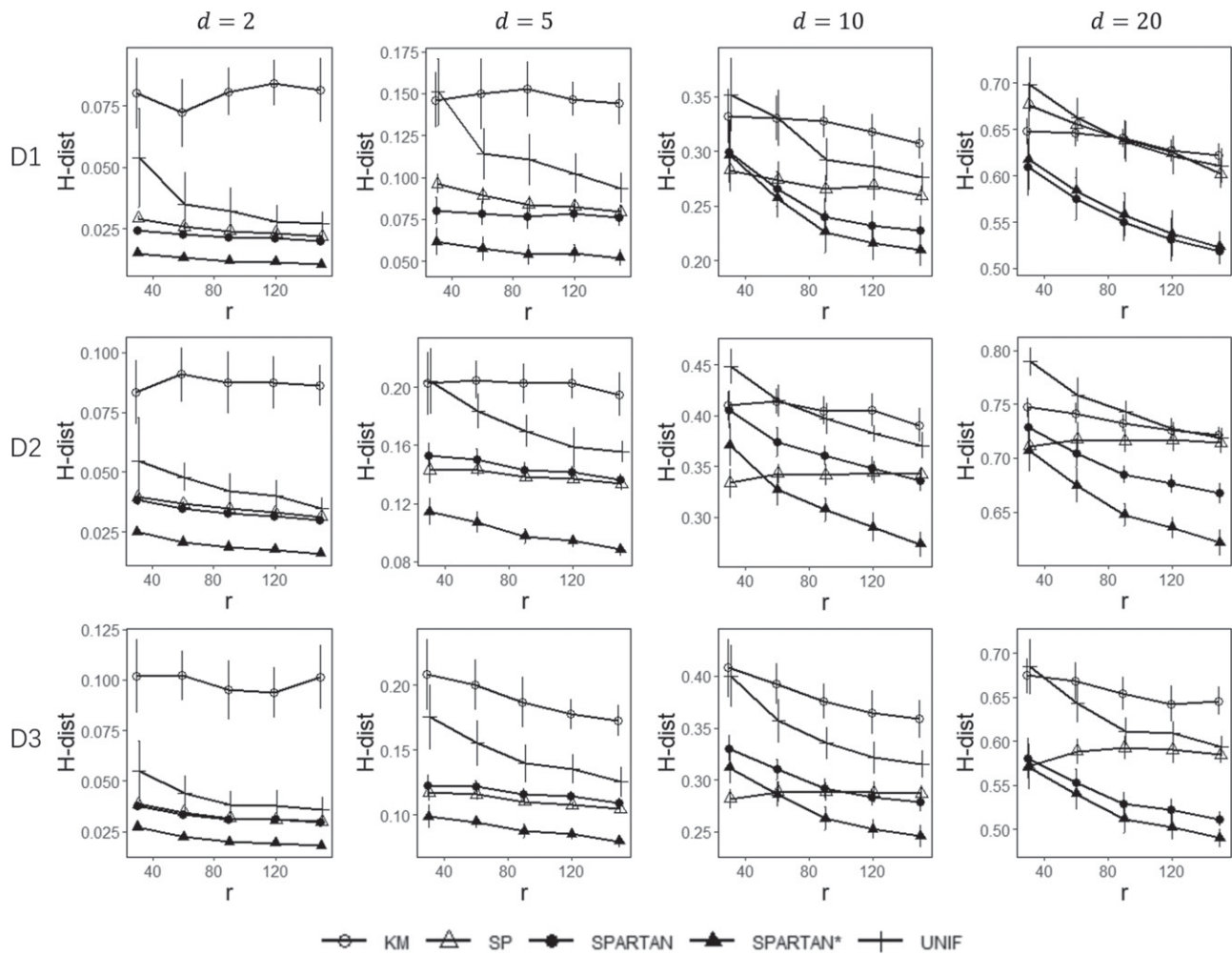
Second, we observe the Hellinger distance yielded by the proposed method decreases as  $r$  increases. Moreover, the proposed method outperforms the uniform subsampling method in all cases. These observations are consistent with **Theorem 1**, which indicates the proposed subsample estimator converges to the true probability density function and is more efficient than the estimator corresponding to the uniform subsampling method.

Third, we observe the proposed estimator with  $\mathbf{H} = r^{-1/(d+4)} \times \hat{\Sigma}^{1/2}$  outperforms the other three competitors in most of the cases. As the same bandwidth matrices are applied in all these estimators, such a comparison is fair. Consequently, the aforementioned observation suggests the subsample identified by the proposed subsampling method is more representative of the observed sample than the subsamples selected by the other three methods. We also observe the proposed estimator with  $\mathbf{H} = r^{-2/(d+6)} \times \hat{\Sigma}^{1/2}$  consistently outperforms the one with  $\mathbf{H} = r^{-1/(d+4)} \times \hat{\Sigma}^{1/2}$ . This observation is consistent with **Theorem 1**, which suggests  $h = O(r^{-2(1-\delta)/(d+6)})$  yields the smallest upper bound of the asymptotic integrated mean squared error for the proposed estimator.

## 6. Real Data Example

### 6.1. Density Estimation

Throughout this section, we consider the banknote authentication dataset, which is extracted from images that were taken



**Figure 3.** Simulation under different  $d$  (from left to right) and different probability density functions (from upper to lower). The Hellinger distance (H-dist) are plotted versus different  $r$ . Vertical bars represent the standard errors.

from 1372 genuine and forged banknotes. Wavelet transform was used to extract four features from images<sup>2</sup>. To evaluate the performance of the proposed subsampling method, we compare it with other competitors in terms of the accuracy of the kernel density estimation and the prediction accuracy in active learning. A brief introduction to active learning will be given later.

We first visualize the banknote authentication dataset and the subsample selected by the proposed method. In Figure 4, the lower diagonal panels show the scatter plots for each pair of the predictors. We select a subsample of size fifty, and the scatter plots for such a subsample are shown in the upper panels of Figure 4. The heat maps are obtained using kernel density estimation. We observe the selected subsample has an appealing visual representation of the original sample.

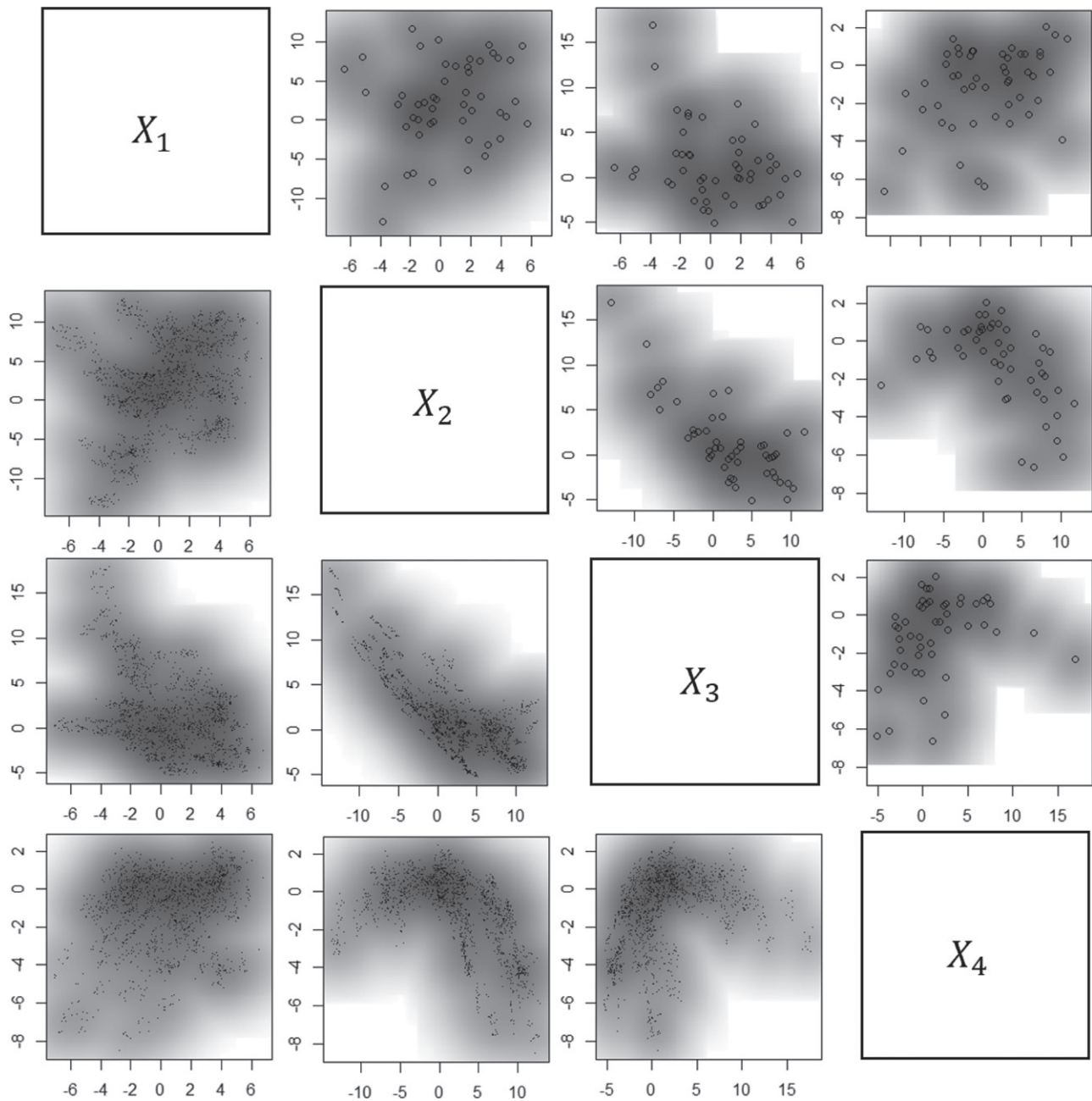
For density estimation, we consider three competitors, as mentioned in the previous section. Same as the settings stated in the previous section, we used the Gaussian kernel for kernel density estimators and the general Scott’s rule to determine the value of the bandwidth matrix. All the parameters are set as the same as the ones we used in the previous section. We

replicated the experiment twenty times. In each replication, the dataset is randomly divided into the training set and the testing set of equal sizes. We first calculate the full sample kernel density estimator using the testing set, denoted by  $\hat{p}_{full}$ . For each subsample kernel density estimator, we then evaluate its estimation accuracy through the empirical Hellinger distance, defined as  $1 - \sum_{i=1}^n \sqrt{\hat{p}(x_i)/\hat{p}_{full}(x_i)}/n$ , where  $\{x_i\}_{i=1}^n$  represents the testing set. This empirical Hellinger distance is not a formal distance and thus may have negative values, as we will see later. Nevertheless, the empirical Hellinger distance can be used as a surrogate for the true Hellinger distance since a small value of the empirical Hellinger distance is associated with a small value of the true Hellinger distance, intuitively.

The left panel of Figure 5 shows the empirical Hellinger distance versus different subsample sizes  $r$ . The standard error bars are obtained from one hundred replicates. We observe that the uniform subsampling method consistently outperforms the K-medoids method. We then observe that the proposed method and the support point method perform similarly, and both have better performance than the uniform subsampling method. Finally, we observe the proposed estimator with  $\mathbf{H} = r^{-2/(d+6)} \times \hat{\Sigma}^{1/2}$ , as guided by Theorem 1, gives the best result. All these observations are consistent with the findings in the previous section.

<sup>2</sup>The dataset can be downloaded from <https://archive.ics.uci.edu/ml/datasets/banknote+authentication>.





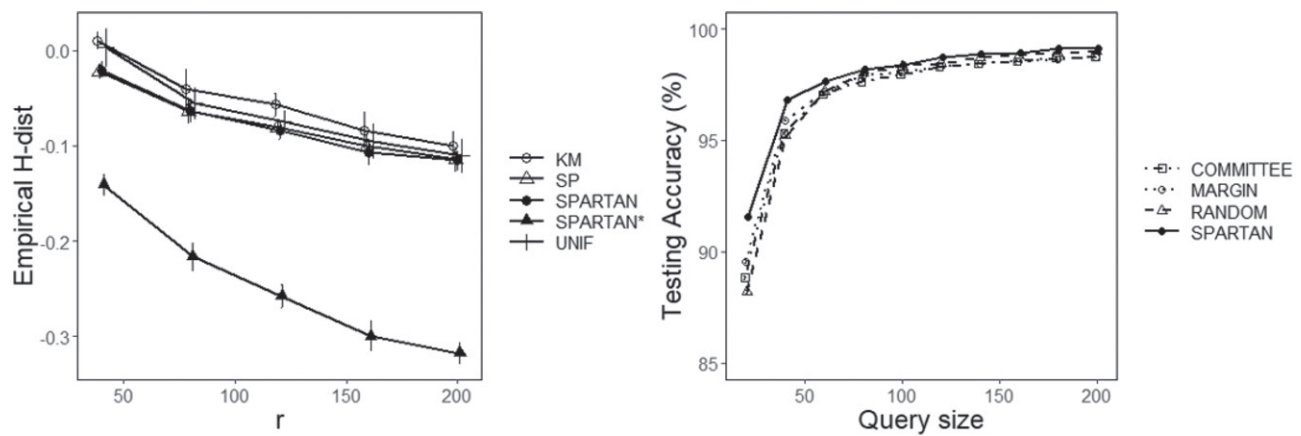
**Figure 4.** Visualization of the banknote authentication dataset. The lower diagonal panels show the scatter plots for each pair of predictors. The upper diagonal panels show the scatter plots for the selected subsample using the proposed algorithm. The heat maps are obtained using kernel density estimation.

## 6.2. Active Learning

We now consider the task of active learning, which aims to make an accurate prediction, with the number of labeled training data points as small as possible (Krogh and Vedelsby 1995; Cohn, Ghahramani, and Jordan 1996). These approaches are essential for numerous sophisticated supervised learning tasks, where the labeled instances are challenging, time-consuming, or expensive to obtain. Take speech recognition as an example; accurate labeling of speech utterances is extremely time-consuming and requires trained linguists. It is reported that annotation at the level of the phoneme can take 400 times longer than the actual audio (Settles 2012). In general, active learning approaches select the data points (also termed as the

query points) iteratively and interactively. In each iteration, one query the oracle to obtain the label at a new query point, based on certain criteria. It is known that a representative subsample is potentially associated with an accurate prediction in active learning (Settles 2012).

The proposed subsampling method can be cast as an active learning approach. In particular, we generate the Sobol sequence (Owen 2003) in Algorithm 1 and select the query points sequentially in Algorithm 2. To evaluate the performance of the proposed method, we compare it with the following baseline methods: (1) random sampling (RANDOM), (2) query by committee (COMMITTEE), which select query points that maximize the disagreement among different models (Settles 2012), and (3)



**Figure 5.** Left: For the density estimation of the banknote authentication dataset, the empirical Hellinger distance (H-dist) is plotted versus different subsample sizes  $r$ . Right: For the active learning of the banknote authentication dataset, the testing accuracy is plotted versus different query sizes. Vertical bars represent the standard errors. In the right panel, the standard errors are tiny, and thus the error bars are almost invisible.

margin-based method (MARGIN) which choose query points that lie on the margin of the decision line (Schohn and Cohn 2000).

We replicate the experiment a hundred times on the banknote dataset. In each replication, the dataset is randomly divided into the training set and the testing set of equal sizes. We evaluate the classification model by its mean classification accuracy on the testing set. The classification accuracy is defined as  $(TP + FN)/n$ , where  $n$  denotes the size of the testing set, and  $TP$  and  $FN$  denote true positive and false negative, respectively. We use the support vector machine, implemented by the R package `e1071` (Meyer et al. 2015), for classification in the active learning. The RBF kernel with default parameters is applied. The size of query points ranges from 10 to 200. For the committee method and the margin-based method, which require several initial labeled data points as input, ten data points are randomly selected and labeled.

The right panel of Figure 5 shows the mean classification accuracy of different active learning methods versus different numbers of query points. The vertical bars represent the standard errors. These bars, however, are almost invisible due to extremely small values of standard errors. We observe the proposed method consistently outperforms all the competitors. We attribute such an observation to the fact that the proposed method selects a representative subsample in a sequential way, resulting in a more accurate prediction in active learning.

## 7. Discussion

In this article, we proposed a novel model-free subsampling method, using the space-filling design and optimal transport techniques. The proposed algorithm is efficient and can be adaptive to the unknown probability density function. Theoretically, we show the proposed subsample kernel density estimator converges to the true probability density function under mild conditions. The order for the optimal smoothing parameter for the proposed kernel density estimator is also derived. The superior performance of the proposed method over mainstream competitors was justified by various numerical experiments.

In this article, we mainly focus on using the unit cube as the target distribution due to mathematical simplicity. In practise, it is possible to consider standard Gaussian distribution instead. Specifically, we could generate the random sample from the standard Gaussian distribution in Algorithm 2, and use the Gaussian Sobol sequence instead of the space-filling design points in Algorithm 1. The other steps remain the same. Empirical results show such a scheme may lead to slightly better performance. The proposed method has the potential to be applied to many large-sample applications, including but not limited to nonparametric regression, kernel methods, and low-rank approximation of matrices. This work may speed up these researches with theoretical guarantees.

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## Conflict of Interest

The authors report there are no competing interests to declare.

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